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Vibration of bilayered cylindrical shells with layers of different materials †

S. H. Arshad[1,*], M. N. Naeem[2], N. Sultana[1], Z. Iqbal[1] and A. G. Shah[3]

[1]*Department of Mathematics, University of Sargodha, Sargodha 40100, Punjab, Pakistan*
[2]*Department of Mathematics, G. C. University Faisalabad, Faisalabad 38000, Punjab, Pakistan*
[3]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Punjab, Pakistan*

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Abstract

In this analysis, a comparative study for natural frequencies of two-layered cylindrical shells was presented with one layer composed of functionally graded material and the other layer of isotropic material. Love's thin shell theory was exploited for the strain-displacement and curvature-displacement relationships. For governing frequency equations, the Rayleigh-Ritz method was utilized to minimize the Lagrangian functional in the form of an eigenvalue problem. Frequency spectra were computed for long, short, thick, and thin cylindrical shells by varying the nondimensional geometrical parameters, length-to-radius and thickness-to-radius ratios for a simply supported end condition. Influence of different configurations of cylindrical shells on the shell frequencies was studied. For validity, the results obtained were compared with some results of isotropic and single-layered functionally graded cylindrical shells from the literature.

Keywords: Functionally graded materials; Isotropic material; Cylindrical shells; Love's thin shell theory; Rayleigh-Ritz method

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1. Introduction

Vibration of shells is one of the extensively studied areas in structural dynamics. Particularly, circular cylindrical shells are widely used in every field of engineering and modern technology. Arnold and Warburton [1] made several maiden attempts to study the frequency response of cylindrical shells. A comprehensive review on vibration of cylindrical shells was given by Leissa [2], who also discussed different shell theories in her monograph. Markus [3] explained many aspects of vibration of cylindrical shells in his collection including bilayered shells.

Shells are structured from isotropic and composite materials. Among composite materials, functionally graded materials (FGMs) have gained great importance due to their mechanical and thermal barrier properties. These properties involve the high temperature resistance, fracture toughness, and strength of materials. They are considered advanced materials for their applications in electronic components, thermal safeguarding system for reusable spacecraft, and blast protection for sensitive structures.

These materials were introduced by Japanese materialists in 1984. Miyamoto et al. [4] compiled their monographs on FGMs. The compilation represents a detailed discussion on the design, processing, application, and thermomechanical properties of these materials. They are fabricated by combining two or more materials by using powder metallurgy technique. Volume fractions of the FGM constituents are graded in the thickness direction and are controlled by power law distribution. Arshad et al. [5] proposed other volume fraction laws for the fabrication of cylindrical shells. Several papers are available on the study of FGM cylindrical shells confined to a single layer [5-7], while only a few studies are on the vibration of multilayered isotropic cylindrical shells. Very few studies are available on sandwich technology. Sofiyev et al. [8] analyzed the vibration and stability analysis of a three-layered conical shell, with the middle layer made up of FGM. Shi-Rong and Batra [9] studied the buckling of a simply supported three-layered circular cylindrical shell under axial compressive load, in which the inner and outer layers of the shell were composed of isotropic materials and the middle layer was composed of FGMs. Markus [3] gave fundamental assumptions for different theories of damped, two-layered cylindrical shells and derived equations of motion based on these theories. No study has been reported on the vibration of cylindrical shells composed of isotropic and FGM layers in the literature.

In the present study, a bilayered cylindrical shell was considered, in which one of the layers was composed of FGM with constituents of nickel (Ni) and stainless steel (SS), while the other layer was made up of isotropic material. In this study, the material for the isotropic layer was one of the FGM con-

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*Corresponding author. Tel.: +92 333 672 7627, Fax.: +92 48 923 0767
E-mail address: shahid26188@yahoo.com

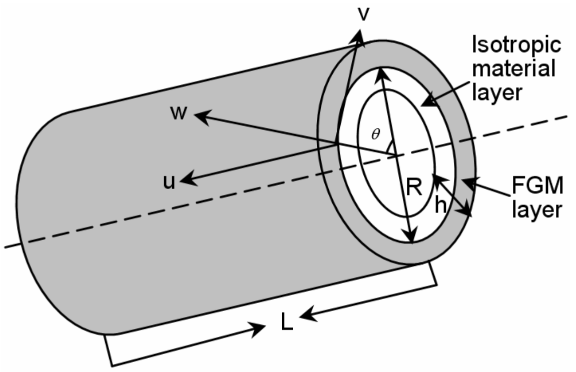

Fig. 1. Geometry of a bilayered cylindrical shell.

stituents. The thickness of both layers was assumed to be equal. Love's thin shell theory was employed for strain-displacement and curvature-displacement relations along with a simply supported end condition. The Rayleigh-Ritz method was utilized to transform the energy equations in eigenvalue form. Shell natural frequencies were tabulated and sketched for long, short, thick, and thin cylindrical shells. For validation, the results of the present approach were compared with those of the isotropic as well as single-layered FGM cylindrical shells from the previous studies.

2. Theoretical formulation

A thin-walled bilayered circular cylindrical shell with constant thickness h, axial length L, and mean radius R was considered for studying its vibration characteristics. An orthogonal coordinate system was established at the middle surface of the shell. u (x, θ, t), v (x, θ, t), and w (x, θ, t) are the midsurface deformation along the axial, circumferential, and radial directions, respectively and t stands for the time variable. The shell was composed of two layers fabricated with different materials. One of the layers was composed of FGM, while the other layer was made up of isotropic material, as shown in Fig. 1. Expressions for strain and kinetic energies of a vibrating bilayered cylindrical shell are given as:

$$S = \frac{1}{2}\int_0^L\int_0^{2\pi}[A_{11}{e_1}^2 + A_{22}{e_2}^2 + A_{12}e_1e_2 + A_{66}{e_{12}}^2 + 2B_{11}e_1\kappa_1 + 2B_{12}(e_1\kappa_2 + e_2\kappa_1) + 2B_{22}e_2\kappa_2 + 2B_{66}e_{12}\kappa_{12} + D_{11}{\kappa_1}^2 + D_{22}{\kappa_2}^2 + 2D_{12}\kappa_1\kappa_2 + 4D_{66}{\kappa_{12}}^2]Rd\theta dx \quad (1)$$

$$K = \frac{1}{2}\int_0^L\int_0^{2\pi}\rho_t[(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2]d\theta dx \quad (2)$$

where e_1, e_2, e_{12} are the reference surface strains, and $\kappa_1, \kappa_2, \kappa_{12}$ are the reference surface curvatures along the normal and share directions, respectively, which are defined in Ref. 5. A_{ij}, B_{ij}, and D_{ij} $(i,j{=}1,2,6)$ are the extensional, coupling, and bending stiffnesses, respectively, and defined in both layers of the shell as:

$$\{A_{ij}, B_{ij}, D_{ij}\} = \underbrace{\int_{-h/2}^{0} Qij^{(1)}\{1, z, z^2\}dz}_{\text{Inner layer material}} + \underbrace{\int_{0}^{+h/2} Qij^{(2)}\{1, z, z^2\}dz}_{\text{Outer layer material}} \quad (3)$$

where the superscripts (1) and (2) represent the inner and outer shell layers, respectively, and Q_{ij} $(i,j{=}1,2,6)$ are the reduced stiffnesses, which are defined in Ref. 5. Dot over deformation displacements in Eq. (2) represent partial differentiation with respect to time variable. ρ_t is the mass density per unit length over both layers and is defined as follows:

$$\rho_t = \int_{-h/2}^{0}\rho^{(1)}dz + \int_{0}^{+h/2}\rho^{(2)}dz \quad (4)$$

The Lagrangian energy functional Π is the difference between the maximum kinetic and strain energies of the vibrating cylindrical shell. It is defined as:

$$\Pi = K_{\max} - S_{\max} \quad (5)$$

In this study, only a simply supported boundary condition was considered at both ends of the cylindrical shells, which is defined as follows:

$$N_x = M_x = v = w = 0; \quad \text{at } x = 0, x = L \quad (6)$$

where N_x and M_x are the forces and moments along the axial direction, respectively. For the numerical solution of the shell problem, the expressions for the general solution for the modal displacements u, v, and w and their nondimensional parameters were taken from Ref. 5. Substituting expressions for strain and kinetic energies along with their modal functions in Eq. (5) and then applying the Rayleigh-Ritz method to externalize the vibration amplitudes A_1, B_1, and C_1 results in a system of three governing eigenvalue equations, which are further transformed into a generalized eigenvalue problem in matrix form as follows:

$$([K] - \Omega^2[M])\bar{X} = 0 \quad (7)$$

where $\Omega^2 = R^2\rho_t\omega^2$ is the eigenvalue corresponding to eigenvectors; and $[K]$ and $[M]$ are the stiffness and mass matrices, respectively, with coefficients C_{ij} and M_{ij} $(i,j{=}1,2,3)$, respectively, $C_{ij}{=}C_{ji}$ and $M_{ij}{=}M_{ji}{=}0$ for $i \neq j$, and $X{=}[A_1\ B_1\ C_1]^T$ (T denotes transpose). Eq. (7) is solved for a nontrivial solution. The MATLAB software was used to solve the eigenvalue problem, which generated a set of three natural frequencies (Hz) of the cylindrical shell; the lowest is of our interest.

3. Functionally graded layers

For a cylindrical shell with two layers, one with isotropic material M_3 and the other composed of FGM with constituents M_1 and M_2, the resulting material parameters Young modulus E, Poisson ratio ν, and mass density ρ of both layers are as follows:

$$E = E_{FGM} + E_{isotropic}$$
$$\nu = \nu_{FGM} + \nu_{isotropic} \quad (8)$$
$$\rho = \rho_{FGM} + \rho_{isotropic}$$

Here, two cases arise. In case I, the FGM layer is placed at the inner surface of the shell, while the isotropic material is at the outer surface. In case II, the FGM layer is at the outer, while the isotropic material layer is inserted into the inner surface of the cylindrical shell. Resultant material properties of the FGM constituents at both FGM layers are given as follows:

$$E^{(1)}{}_{FGM} = (E_2 - E_1)\left(\frac{2z}{h}+1\right)^p + E_1$$
$$\nu^{(1)}{}_{FGM} = (\nu_2 - \nu_1)\left(\frac{2z}{h}+1\right)^p + \nu_1 \quad (9)$$
$$\rho^{(1)}{}_{FGM} = (\rho_2 - \rho_1)\left(\frac{2z}{h}+1\right)^p + \rho_1$$

$$E^{(2)}{}_{FGM} = (E_2 - E_1)\left(\frac{2z}{h}\right)^p + E_1$$
$$\nu^{(2)}{}_{FGM} = (\nu_2 - \nu_1)\left(\frac{2z}{h}\right)^p + \nu_1 \quad (10)$$
$$\rho^{(2)}{}_{FGM} = (\rho_2 - \rho_1)\left(\frac{2z}{h}\right)^p + \rho_1$$

where E_1, ν_1, ρ_1 and E_2, ν_2, ρ_2 are the material properties of the FGM constituents M_1 and M_2, respectively, and p is the power law exponent, $0 \le p \le \infty$, which controls the volume fractions of the FGM constituents.

Figs. 2(a)-2(d) show the possible configurations for the fabrication of the cylindrical shells. Four types of shells were generated by altering the order of the materials in the layers and by varying the configuration of the FGM constituents in the FGM layer. The FGM layer was composed of two constituents, Ni and SS, while at the isotropic layer, any isotropic material helpful in maximizing the mechanical properties of the shell can be applied. In type I and III shells, Ni was taken at the inner side, and SS was taken at the outer side in the FGM layer; in type II and IV shells, this order was reversed. The material properties given in Eq. (9) are for the inner FGM layer of type I and II shells, which vary from $-h/2$ to 0, whereas the properties defined in Eq. (10) are for the outer FGM layer of the shell types III and IV, which vary from 0 to $+h/2$. From these relations, when $z = -h/2$, the resultant material properties were $E = E_1$, $\nu = \nu_1$, and $\rho = \rho_1$; when $z = 0$, these properties were reduced to $E = E_2$ and E_3, $\nu = \nu_2$, and ν_3, and $\rho = \rho_2$ and ρ_3, and when $z = +h/2$, these became $E = E_3$, $\nu = \nu_3$, and $\rho = \rho_3$, respectively, for shell type I. Resultant material properties for shells II–IV can be obtained in a similar fashion.

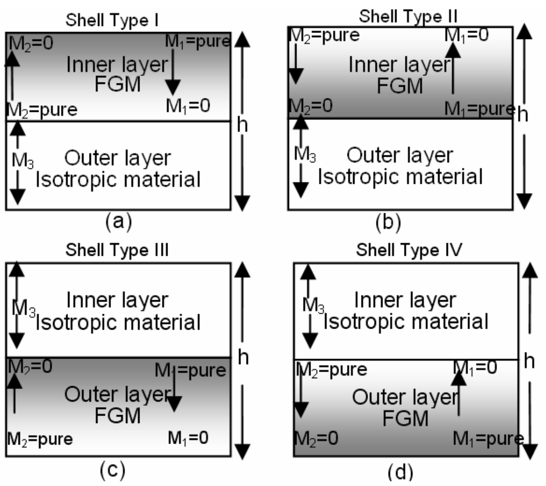

Fig. 2. Types of bilayered cylindrical shells.

3.1 Variation of volume fraction of the FGM constituents in the FGM layer

In Fig. 3, variation of volume fractions V_1 and V_2 for the constituent materials M_1 and M_2 of the FGM layer of the cylindrical shell are sketched against the intrinsic thickness variable z/h. In case I, when the FGM layer was kept at the inner side of the shell, the volume fraction V_1 of material M_1 increased from 0 to 1 as z/h went from –0.5 to 0, while the volume fraction V_2 of material M_2 decreased from 1 to 0 as z/h varied from –0.5 to 0 in shell types I and II, respectively. Similarly, in case II, when the FGM layer was inserted into the outer side of the shell, V_1 of material M_1 increased from 0 to 1 as z/h moved from 0 to 0.5, while the volume fraction V_2 of material M_2 decreased from 1 to 0 as z/h varied from 0 to 0.5 in shell types III and IV, respectively. This increment in the volume fractions V_1 of materials M_1 and decrement in the volume fraction V_2 of material M_2 depended on the values of radial variable z/h and power law exponent p. Moreover, in case I, as $-0.50<z/h<-0.25$ and $p<1$, volume fraction V_1 increased rapidly; as $-0.25<z/h<0$ and $p<1$, V_1 increased slowly; as $-0.50<z/h<-0.25$ and $p>1$, V_1 increased sluggishly; but for $p>1$ and $-0.25<z/h<0$, V_1 increased briskly. Hence, in all the cases, V_1 approached its maximum value of 1 at the shell midsurface. In the same way, V_2 decreased rapidly as $-0.50<z/h<-0.25$ and $p<1$, while it decreased slowly as $-0.25<z/h<0$ and $p<1$. V_2 decreased slowly as $-0.50<z/h<-$

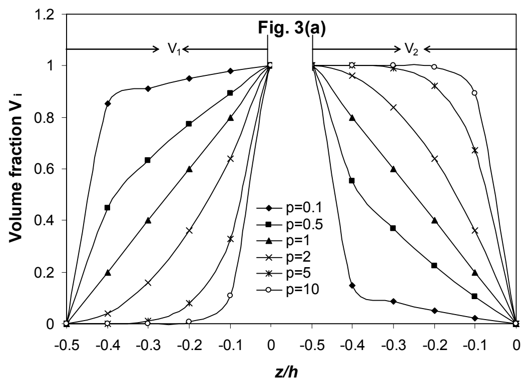

Fig. 3. Variation of volume fractions V_1 and V_2 of constituent materials M_1 and M_2 at the inner FGM layer (case I).

0.25 and p>1, while it decreased swiftly as p>1 and –0.25<z/h<0. Eventually, V_2 approached its minimum value of 0 at the inner shell surface of the inner FGM layer shown in Fig. 3. In case II, for the outer FGM layer, similar behavior of volume fractions V_1 and V_2 for the constituent materials M_1 and M_2 was observed. However, in this case, the spans of the intrinsic variable were 0<z/h<+0.25 and +0.25<z/h<+0.50 under the same values of power law exponent p.

4. Results and discussions

For validity, the results of the present approach were compared with the corresponding results of isotropic and single-layered FGM cylindrical shells available in the literature.

In Table 1, results of the variation of frequency parameters $\Omega=\omega R\sqrt{((1-v^2)\rho/E)}$ are compared for circumferential wave number n with the available three-dimensional literature results of Markus [3] for shell parameters (m=1, L/R=20, h/R = 0.05, 0.002, v=0.3) for a simply supported cylindrical shell. As the configuration of the FGM layer of shell types I and III is identical to the first configuration of Loy et al. [6], similarly, the configuration of shell types II and IV resembles their second configuration. Therefore, natural frequencies (Hz) of the bilayered cylindrical shells were compared with the single-layered FGM cylindrical shell with the same configuration as in Loy et al. [6] in Table 2. It is evident from Tables 1 and 2 that the results of the present approach are in good agreement with those of the previous studies.

In Fig. 4, the variation of natural frequencies (Hz) of the isotropic, single-layered FGM, bilayered cylin drical shells with the layers of isotropic materials and FGMs are sketched with circumferential wave number n for the shell parameters (m=1, L/R=20, , h/R=0.002, v=0.3). Constituent materials for the FGM layers were Ni and SS, while the material for the isotropic layer was taken from the FGM constituents. It is clear from these figures that natural frequencies (Hz) first decrease up to a certain value of circumferential wave number $n=3$ and then increase smoothly with the increasing values of circumferential wave number n. Moreover, the natural frequencies of the single-layered FGM cylindrical shell and the bilayered cylindrical shells lay in between the natural frequencies of the isotropic cylindrical shells. The natural frequencies of bilayered cylindrical shells with SS at the isotropic layer were greater than those of the bilayered cylindrical shell Ni at the isotropic layer. In Figs. 5(a) and 5(b), the dotted lines represent the natural frequencies of thick and short shells, while the continuous lines correspond to the thin and long cylindrical shells, respectively. In Fig. 5(a), variation of the natural frequencies (Hz) of bilayered shells is drawn against circumferential wave number n for axial half wave number m=1, 2, 3, 4, 5 at power law exponent p=5 for thin shell (h/R=0.002) and for thick shell (h/R=0.01), respectively, at L/R=20. In Fig. 5(b), the same behavior of the natural frequencies is observed for long (L/R=20) and short (L/R=5) cylindrical shells. It is noted from these figures that the natural frequencies of the thick shells are higher than those of thin shells, and frequencies of the short shells are higher than those of the long shells. Moreover, the frequencies of higher axial modes are higher than those of the lower axial modes.

In Table 3, the variation of minimum natural frequencies are tabulated against L/R=1, 5, 20 with the constant thickness-to-radius ratio h/R=0.002 at p=0.5, 1, 5 for the four types of bilayered cylindrical shells. It is evident from these results that the minimum natural frequencies (Hz) occur at circumferen tial wave numbers 11, 5, and 3, respectively. The behavior of natural frequencies of shell types I and III is nearly the same as in both shells, the natural frequencies increase with the increasing value of p.

Table 1. Comparison of the frequency parameters.

h/R	n	Three dimensional [3]	Present
0.05	0	0.0929296	0.092968
	1	0.0161063	0.016103
	2	0.0392332	0.039271
	3	0.109477	0.10981
	4	0.209008	0.21028
0.002	0	0.0929296	0.092930
	1	0.0161011	0.0161010
	2	0.00545243	0.0054530
	3	0.00503724	0.0050415
	4	0.00853409	0.0085338

Table 2. Comparison of the natural frequencies (Hz) of bilayered cylindrical shells with layers of different materials.

n	Ref. 6	Shell type		Ref..6	Shell type	
		I	III		II	IV
1	13.433	13.490	13.490	12.998	13.266	13.266
2	4.5504	4.5722	4.5701	4.4068	4.4991	4.4924
3	4.2191	4.2461	4.2387	4.0891	4.1737	4.1685
4	7.1510	7.1971	7.1843	6.9251	7.0650	7.0696
5	11.425	11.498	11.478	11.061	11.283	11.297

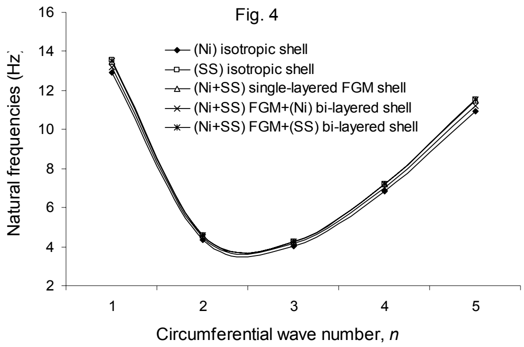

Fig. 4. Variation of the natural frequencies (Hz) with circumferential wave number n of isotropic, single-layered FGM, and bilayered cylindrical shells with shell parameters m=1, v=0.3, L/R=20, h/R=0.002, p=10.

Table 3. Variation of the minimum natural frequencies (Hz) with length-to-radius ratios L/R for h/R=0.002 and m=1 for type I–IV cylindrical shells for the different values of p.

Shell type	L/R	$Ni^{p=0}$	$SS^{p=0}$	p=0.5	p=1	p=5
I	1	82.975	87.313	85.931	86.280	86.969
	5	16.076	16.914	16.646	16.714	16.847
	20	4.0481	4.2625	4.1945	4.2122	4.2461
II	1	82.975	87.313	86.579	86.226	85.552
	5	16.076	16.914	16.773	16.705	16.575
	20	4.0481	4.2625	4.2247	4.2069	4.1737
III	1	82.975	87.313	85.647	86.025	86.843
	5	16.076	16.914	16.593	16.666	16.824
	20	4.0481	4.2625	4.1802	4.1985	4.2387
IV	1	82.975	87.313	86.561	86.179	85.376
	5	16.076	16.914	16.768	16.694	16.539
	20	4.0481	4.2625	4.2264	4.2080	4.1685

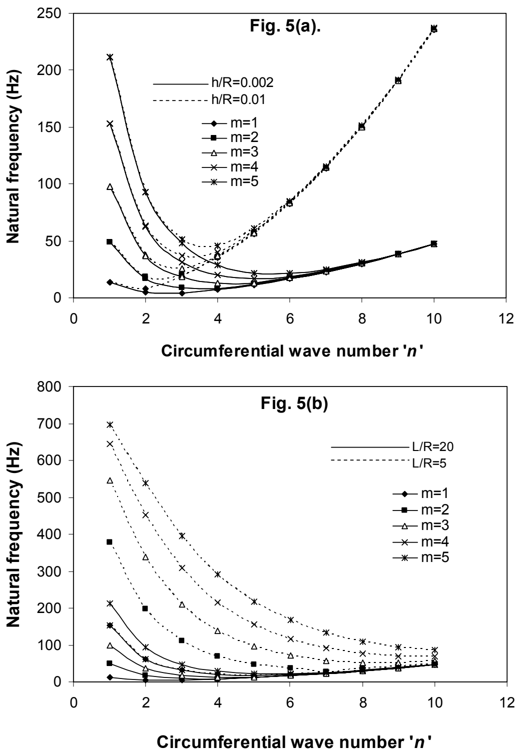

Fig. 5. Variation of the natural frequencies (Hz) of bilayered cylindrical shell type II with circumferential wave number n (a) for thick and thin shells and (b) for long and short shells.

It also increase for the greater value of the L/R ratio. The same response of minimum natural frequencies for shell types II and IV is observed, but in this case, the minimum natural frequencies of the cylindrical shells decrease with the increas ing value of power law exponent p. Similar behavior of the frequncies is observed when minimum natural frequencies of the shells are tabulated with the h/R ratio, as shown in Table 4 but this time minimum natural frequencies turn out at circumferential wave numbers 3, 2, 1 for corresponding values of h/R ratios 0.001, 0.01, 0.05 respectively, for shells type I-IV. Entries in columns $Ni^{p=0}$ and $SS^{p=0}$ are for the natural frequencies of Ni and SS isotropic cylindrical shells, respectively.

Table 4. Variation of minimum natural frequencies (Hz) with thickness-to-radius ratios h/R for L/R=20 and m=1 for type I–IV cylindrical shells for different values of p

Shell type	h/R	$Ni^{p=0}$	$SS^{p=0}$	p=0.5	p=1	p=5
I	0.001	2.6535	2.7916	2.7474	2.7585	2.7805
	0.01	7.5279	7.9251	7.7991	7.8316	7.8944
	0.05	12.896	13.549	13.323	13.379	13.492
II	0.001	2.6535	2.7916	2.7684	2.7572	2.7357
	0.01	7.5279	7.9251	7.8560	7.8232	7.7618
	0.05	12.896	13.549	13.435	13.379	13.268
III	0.001	2.6535	2.7916	2.7387	2.7508	2.7768
	0.01	7.5279	7.9251	7.7724	7.8065	7.8812
	0.05	12.896	13.549	13.322	13.378	13.491
IV	0.001	2.6535	2.7916	2.7675	2.7552	2.7297
	0.01	7.5279	7.9251	7.8576	7.8232	7.7499
	0.05	12.896	13.549	13.435	13.378	13.266

5. Conclusions

This study shows that when two different materials are inserted into the layers of bilayered cylindrical shells, their natural frequencies (Hz) are affected very slightly for the same configuration of FGM constituents in the inner and outer FGM layers. If the order of the configuration is reversed, then the influence on the natural frequencies of the shells is considerable. This work can be extended for studying other boundary conditions, mode shapes, and buckling and postbuckling phenomena of cylindrical shells.

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References

[1] R. N. Arnold and G. B. Warburton, The flexural vibrations of thin cylinders, *Proc. Instn. Mech. Engrs.,* A167 (1953) 62-88.

[2] A. W. Leissa, *Vibration of shells*, NASA SP-288, (1997), Reprinted by Acoustical Society of America, America institute of Physics, (1993).

[3] S. Markus, *The mechanics of vibration of cylindrical shells*, Amsterdam, Elsevier, (1988).

[4] Y. Miyamoto, W. A. Kaysser, B. H. Rabin, A. Kawasaki and R. Ford, *Functionally graded materials: design, processing and application*, Kulewer Academic Publishers, London, (1999).

[5] S. H. Arshad, M. N. Naeem and N. Sultana, Frequency analysis of functionally graded material cylindrical shells with various volume fraction laws, *J. Mech. Engs. Sc.,* 221 (C-12) (2007) 1483-1495.

[6] C. T. Loy, K. M. Lam and J. N. Reddy, Vibration of functionally graded cylindrical shells, *Int. J. Mech. Sc.,* 41 (1999) 309-324.

[7] S. C. Pradhan, C. T. Loy, K. M. Lam and J. N. Reddy, Vibration characteristics of functionally graded cylindrical shells under various boundary conditions, *Appl. Accoust*., 61 (2000) 111-129.

[8] A. H. Sofiyev, A. Deniz, I. H. Akeay and E. Yusufoglu, The vibration and stability of a three-layered conical shell containing FGM layer subject to axial compression load, *Acta Mech*., 183 (2006) 129-144.

[9] Shi-Rong Li and R. C. Batra, Buckling of axially compressed thin cylindrical shells with functionally graded middle layer, *Thin-Walled strct*., 44 (2006) 1039-1047.

Shahid Hussain Arshad obtained his master degree in Mathematics from the University of the Punjab, Pakistan in 1997. Presently, he is undertaking his Ph.D. studies in the Department of Mathematics, University of Sargodha, Pakistan. He has also served as a visiting research scholar in the Department of Engineering Science and Mechaincs, Virginia Tech, and State University, Virginia, USA in 2007-2008. His research interests are structural dynamics and material science.